MTH 265. Week 4 Monday Lecture Notes.
Taylor Series + Applications.

Definition 1. Let $f(x)$ be an infinitely differentiable function on some interval containing the base point $x=a$;

Recall that ① The n^{th} order Taylor polynomial $T_n(x)$ is the sum $T_n(x) = \sum_{n=0}^{N} \frac{f^{(k)}(a)}{k!}(x-a)^k$;

② The n^{th} order Remainder term $R_n(x)$ is defined as $R_N(x) = f(x) - T_n(x)$;

Then, The Taylor Series $T(x)$ of $f(x)$ is the series $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$;

If $a=0$, $T(x)$ is sometimes called the Maclaurin Series of $f(x)$;

Example 1.1. Determine the Taylor series $T(x)$ of $f(x)=e^x$ about $a=0$;

Since $f^{(n)}(x) = \frac{d^n}{dx^n}(e^x) = e^x$ and $f^{(n)}(0) = e^0 = 1$ $\therefore$ $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$;

Example 1.2. Determine the Taylor series $T(x)$ of $f(x) = \ln(1-x)$ about $a=0$;

The n^{th} derivatives of $f(x)$:

$$f^{(1)}(x) = (1-x)^{-1}(-1);$$
$$f^{(2)}(x) = (-1)(1-x)^{-2}(-1)(-1) = (1-x)^{-2}(-1);$$
$$f^{(3)}(x) = (-2)(1-x)^{-3}(-1)(-1) = (-2)(1-x)^{-3};$$
$$f^{(4)}(x) = (-2)(-3)(1-x)^{-4}(-1) = (-1)(3!)(1-x)^{-4};$$
$$f^{(5)}(x) = (-1)(3!)(-4)(1-x)^{-5}(-1) = (-1)(4!)(1-x)^{-5};$$
$$\vdots$$
$$f^{(n)}(x) = (-1)(n-1)!\,(1-x)^{-n} \quad \text{for } n \geq 1;$$

{ This is typically shown using induction but pattern recognition is fine for this course;

Then, $f^{(n)}(x) = \begin{cases} \ln(1-x) & \text{if } n=0 \\ (-1)(n-1)!\,(1-x)^{-n} & \text{if } n \geq 1 \end{cases}$; and

$$f^{(n)}(0) = \begin{cases} \ln(1) = 0 & \text{if } n=0 \\ (-1)(n-1)!\,(1)^{-n} = (-1)(n-1)! & \text{if } n \geq 1 \end{cases};$$

So, $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \cancel{\frac{f^{(0)}(0)}{0!}x^0}^{\,0} + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=1}^{\infty} \frac{(-1)(n-1)!}{n!}x^n = \boxed{\sum_{n=1}^{\infty} \frac{(-1)}{n}x^n}$;

Example 1.3. Find the Taylor series $T(x)$ of $f(x) = \cos(x)$ about $a=0$;

$$f^{(n)}(x) = \begin{cases} \cos(x) & n=4k \\ -\sin(x) & n=4k+1 \\ -\cos(x) & n=4k+2 \\ \sin(x) & n=4k+3 \end{cases} \text{ if for some } k \in \mathbb{Z}; \text{ Then, } f^{(n)}(0) = \begin{cases} \cos(0) = 1 & n=4k \\ -\sin(0) = 0 & n=4k+1 \\ -\cos(0) = -1 & n=4k+2 \\ \sin(0) = 0 & n=4k+3 \end{cases} \text{ if};$$

Observe that if n is odd, $f^{(n)}(0) = 0$; Reindexing $T(x)$:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{m=0}^{\infty}\left[\frac{f^{(2m)}(0)}{(2m)!}x^{2m} + \cancel{\frac{f^{(2m+1)}(0)}{(2m+1)!}}^{\,0}x^{2m+1}\right] = \sum_{m=0}^{\infty} \frac{f^{(2m)}(0)}{(2m)!}x^{2m};$$

If $m=2k$, i.e. m is even: $2m = 2(2k) = 4k$ and $f^{(2m)}(0) = 1 = (-1)^m$;

If $m=2k+1$, i.e. m is odd: $2m = 2(2k+1) = 4k+2$ and $f^{(2m)}(0) = -1 = (-1)^m$;

Finally, $T(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \overset{\text{relabeling}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$;

Proposition 2. Let $f(x) = T_n(x) + R_n(x)$ where $f(x)$ is infinitely differentiable on $(a-R, a+R)$ for some $a \in \mathbb{R}$, $R > 0$ and $T_n(x)$ is the n^{th} order Taylor polynomial of $f(x)$ about $x=a$;

If $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in (a-R, a+R)$,

then $f(x) = T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ where $T(x)$ is the Taylor series of $f(x)$ about $x=a$;

☆ i.e. the Taylor series of $f(x)$ is a power series representation of $f(x)$ on $(a-R, a+R)$;

Taylor's Inequality (restated).

Let $f(x)$ be infinitely differentiable on some interval containing $(a-R, a+R)$ and let $T_n(x)$ be the n^{th} order Taylor polynomial of $f(x)$ about $x=a$;

If $|f^{(n+1)}(x)| \le M$ on $[a-R, a+R]$, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ on $[a-R, a+R]$;

Remark: This is typically used with the Squeeze Theorem.

Lemma. For any $x \in \mathbb{R}$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$;

Lemma. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$ for any sequence (a_n);

Example 2.1. From Example 1.1: $f(x) = e^x$ has the Taylor series $T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; Find all $x \in \mathbb{R}$ such that $f(x) = T(x)$;

Fix $d \in \mathbb{R}$. We want to apply Proposition 2 on $(-d, d)$;

Since for all $n \ge 0$: $f^{(n)}(x) = e^x$ and $f^{(n)}(x)$ is increasing, choose $M = e^d$.

Then, for all $x \in (-d, d)$: $f^{(n)}(x) = e^x \le M = e^d$;

By Taylor's inequality, $|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}$;

By the Squeeze Theorem, $0 \le \lim_{n\to\infty} |R_n(x)| < \lim_{n\to\infty} \frac{e^d|x|^{n+1}}{(n+1)!} = 0$ since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x.

$\therefore \lim_{n\to\infty} R_n(x) = 0$ on $(-d, d)$ and by Proposition 2, $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $x \in (-d, d)$;

Since $d \in \mathbb{R}$ is chosen arbitrarily, $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $\boxed{x \in \mathbb{R}}$;

Example 2.2. From Example 3.3: $f(x) = \cos(x)$ has the Taylor series $T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$;

We want to apply Proposition 2 on $\mathbb{R}$.

From earlier, $f^{(n)}(x) = \begin{cases} \cos(x) & n = 4k \\ -\sin(x) & \text{if } n = 4k+1 \\ -\cos(x) & n = 4k+2 \\ \sin(x) & n = 4k+3 \end{cases}$; So, we can choose $M = 1$ since $|f^{(n)}(x)| \le 1 = M$ for all $x \in \mathbb{R}$.

By Taylor's Inequality, $|R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1} = \frac{|x|^{n+1}}{(n+1)!}$ for all $x \in \mathbb{R}$;

Then, $0 \leq \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$; $\lim_{n\to\infty} |R_n(x)| = 0$ and $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in \mathbb{R}$.

By Proposition 2, $f(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Theorem 3. Power Series Representations of Selected Functions.

(1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with $R = 1$;

(2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with $R = \infty$;

(3) $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ with $R = \infty$;

(4) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ with $R = \infty$;

(5) $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ with $R = 1$;

(6) $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ with $R = 1$;

Example 3.1. Evaluate $\int_0^1 e^{-x^2}\,dx$ accurate to 6 decimal places.

Part (1). Find a p.s.r. for the integral. Let $f(x) = e^x$ and let $I = \int_0^1 e^{-x^2}\,dx = \int_0^1 f(-x^2)\,dx$;

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$ and $[0,1] \subseteq \mathbb{R}$,

$$e^{-x^2} = f(-x^2) = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}; \text{ let } F(x) = \int e^{-x^2}\,dx;$$

Then, $F(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$;

Finally, $I = \int_0^1 e^{-x^2}\,dx = F(1) - F(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} (1)^{2n+1} - (0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}$;

Part (2). Use the Alternating Series Estimation Theorem to find the minimum number of terms.

Let $b_n = \frac{1}{n!(2n+1)}$; Check that I converges by the Alternating Series Test;

(1) b_n is positive for all $n \geq 0$;

(2) $b_{n+1} = \frac{1}{(n+1)!(2n+3)} < \frac{1}{n!(2n+1)} = b_n$ since $\frac{1}{(n+1)(2n+3)} < \frac{1}{(2n+1)}$ for $n \geq 0$;

(3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!(2n+1)} = 0$;

$\therefore$ We can use the Alternating Series Estimation Theorem: $|S - S_N| < b_{N+1}$ for $N \geq 0$;

We want to find $N \geq 0$ such that $b_{N+1} < \frac{1}{2}(10^{-6})$ for 6 decimal places.

Equivalently, $\frac{1}{(n+1)!(2n+3)} < \frac{1}{2}(10^{-6}) = 5.0 \times 10^{-7} \iff (n+1)!(2n+3) > 2\,000\,000$;

By brute force: For $N = 6$: $685\,440 \not> 2\,000\,000$;

For $N = 7$: $6\,894\,720 > 2\,000\,000$; ✓

Alternatively, For $N=6$: $b_7 = 1.46 \times 10^{-6} \nless 5.0 \times 10^{-7}$;
For $N=7$; $b_8 = 1.45 \times 10^{-7} < 5.0 \times 10^{-7}$; ✓

$\therefore$ We need up to the index $N=7$ to be accurate within 6 decimal places.

Part (3). Calculate the approximation.

$$S_7 = \sum_{n=0}^{7} \frac{(-1)^n}{n!\,(2n+1)} \approx 0.746\,823, \text{ rounded to 6 decimal places.}$$

$$\therefore I = \int_0^1 e^{-x^2}\,dx \approx \boxed{0.746\,823};$$

Example 3.2. Find an approximation of $I = \int_0^{\frac{1}{2}} \sin(x^2)\,dx$ accurate to within 9 decimal places.

Part (1). Find a p.s.r. for $F(x) = \int \sin(x^2)\,dx$ over $[0, \frac{1}{2}]$;

Since $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ on $\mathbb{R}$ and $[0, \frac{1}{2}] \subseteq \mathbb{R}$,

$$F(x) = \int \sin(x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{4n+2}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)};$$

Then, $I = \int_0^{\frac{1}{2}} \sin(x^2)\,dx = F(\frac{1}{2}) - F(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+3)} \left(\frac{1}{2}\right)^{4n+3}$;

Part (2). Find the minimum number of terms needed.

Observe that $F(\frac{1}{2})$ is an alternating series. WTS $F(\frac{1}{2})$ converges by AST so we can use the estimation theorem.

Let $b_n = \frac{1}{(2n+1)!\,(4n+3)\,2^{4n+3}}$;

(1) $b_n > 0$ for all $n \geq 0$;

(2) $b_{n+1} = \frac{1}{(2n+3)!\,(4n+7)(2)^{4n+7}} < \frac{1}{(2n+1)!\,(4n+3)(2)^{4n+3}} = b_n$ for $n \geq 0$ since $(2n+3)(2n+2)(2)^4 > 1$ and $4n+7 > 4n+3$;

(3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(2n+1)!\,(4n+3)\,2^{4n+3}} = 0$;

$\therefore F(\frac{1}{2})$ converges by AST and we can use the estimation theorem: $|S - S_N| < b_{N+1}$ for $N \geq 0$;

For 9 decimal places, we want to find $N \geq 0$ such that $b_{N+1} < \frac{1}{2}(10^{-9}) = 5 \times 10^{-10}$;

By brute force: For $N=1$: $b_2 = 3.70 \times 10^{-7}$;
For $N=2$: $b_3 = 4.03 \times 10^{-10}$; This is enough;

Part (3). Use a calculator to get the approximation;

$$I \approx S_2 = \sum_{n=0}^{2} \frac{1}{(2n+1)!\,(4n+3)} \left(\frac{1}{2}\right)^{4n+3} = \boxed{0.041\,481\,025} \text{ rounded to 9 decimal places.}$$